axioms

Article

Exploring Stochastic Heat Equations: A Numerical Analysis with Fast Discrete Fourier Transform Techniques

Ahmed G. Khattab [1], Mourad S. Semary [1,2], Doaa A. Hammad [1] and Aisha F. Fareed [1,3,*]

1 Basic Engineering Sciences Department, Benha Faculty of Engineering, Benha University, Benha 13512, Egypt; ahmed.khttab@bhit.bu.edu.eg (A.G.K.); mourad.semary@bhit.bu.edu.eg or morad.samir@buc.edu.eg (M.S.S.); doaa.hammad@bhit.bu.edu.eg (D.A.H.)
2 Basic Sciences Department, Faculty of Engineering, Badr University in Cairo BUC, Cairo 11829, Egypt
3 Department of Electrical Engineering, College of Engineering, Prince Sattam bin Abdulaziz University, Al Kharj 11942, Saudi Arabia
* Correspondence: aisha.farid@yahoo.com or a.fareed@psau.edu.sa

Abstract: This paper presents an innovative numerical technique for specific classes of stochastic heat equations. Our approach uniquely combines a sixth-order compact finite difference algorithm with fast discrete Fourier transforms. While traditional discrete sine transforms are effective for approximating second-order derivatives, they are inadequate for first-order derivatives. To address this limitation, we introduce an innovative variant based on exponential transforms. This method is rigorously evaluated on two forms of stochastic heat equations, and the solutions are compared with those obtained using the established stochastic ten non-polynomial cubic-spline method. The results confirm the accuracy and applicability of our proposed method, highlighting its potential to enhance the numerical treatment of stochastic heat equations.

Keywords: compact finite difference; fast discrete Fourier transforms; stochastic heat equations; stochastic partial differential equations; white noise

MSC: 65C30; 65L12

Citation: Khattab, A.G.; Semary, M.S.; Hammad, D.A.; Fareed, A.F. Exploring Stochastic Heat Equations: A Numerical Analysis with Fast Discrete Fourier Transform Techniques. *Axioms* **2024**, *13*, 886. https://doi.org/10.3390/axioms13120886

Academic Editors: Shusen Pu and Subhash Bagui

Received: 16 November 2024
Revised: 18 December 2024
Accepted: 19 December 2024
Published: 21 December 2024

1. Introduction

The classical heat equation [1,2] shows how heat conduction is distributed in a rod, or each region over a given time. It is of fundamental importance in diverse scientific fields. In mathematics, it is used in the prototypical parabolic partial differential equation (PDE). In probability theory, the heat equation relates to the study of Brownian motion via the Fokker–Planck equation. In financial mathematics, it is used to solve the Black–Scholes PDE. The diffusion equation, a more general version of the heat equation, arises in connection with the study of chemical diffusion and other related processes.

The stochastic heat equation (SHE) is also a mathematical model that characterizes the distribution of heat in a medium over time, influenced by random fluctuations or noise. Unlike the classical heat equation, which is deterministic, the stochastic version includes a random forcing term, representing the inherent uncertainties in real-world systems, such as thermal noise. Moreover, these stochastic terms are essential for capturing the effects of randomness, which are important in various physical and mathematical systems, such as those affected by thermal fluctuations or other unpredictable forces. The stochastic heat equation combines aspects of partial differential equations with stochastic processes and provides an important tool in different fields like engineering, physics, finance, and biology [3]. The inclusion of the stochastic term makes solving the stochastic partial differential equations analytically much more complex. Therefore, numerical algorithms will be necessary to investigate their approximate solutions such as the ten non-polynomial cubic-spline (TNPCS) method [4,5], the Chebyshev wavelets with the Galerkin method [6],